International Mathematics Research Notices Advance Access published April 22, 2009
Keller, B., and W. Lowen. (2009) "On Hochschild Cohomology and Morita Deformations," International Mathematics Research Notices, Article ID rnp050, 15 pages.
doi:10.1093/imrn/rnp050

On Hochschild Cohomology and Morita Deformations

Bernhard Keller[1] **and Wendy Lowen**[2]

[1]UFR de Mathématiques, Université Denis Diderot – Paris 7, Institut de Mathématiques, UMR 7586 du CNRS, 2, Place Jussieu, 75251 Paris Cedex 05, France and [2]Departement Wiskunde–Informatica, Universiteit Antwerpen, Middelheimlaan 1, 2020 Antwerpen, Belgium

Correspondence to be sent to: keller@math.jussieu.fr

In this paper we show that, in general, first-order Morita deformations are too limited to capture the second Hochschild cohomology of a differential graded category. For differential graded categories with bounded above cohomology, the Morita deformations do constitute a *part* of the Hochschild cohomology.

1 Introduction

It is a general philosophy that the Hochschild complex of a mathematical object governs its deformation theory and that, in particular, the second Hochschild cohomology group parametrizes its first-order deformations. This, of course, holds true for associative algebras [3], and more generally for schemes and abelian categories ([9], see also [1]). From the inspection of the Hochschild complex of a dg algebra A, it follows that a Hochschild two-cocycle ϕ naturally determines a first-order deformation $A_\phi[\epsilon]$ of A which is not necessarily a dg algebra, not even an A_∞-algebra, but rather an $A_{[0,\infty[}$-algebra or *curved* A_∞-algebra (see e.g. [10]). In particular, the object $A_\phi[\epsilon]$ is equipped with a differential d_ϕ, satisfying $d_\phi^2 = [\phi_0\epsilon, -]$. In this paper, we investigate to which extent the second Hochschild cohomology parametrizes first-order *Morita deformations* of A, i.e. $k[\epsilon]$-linear dg algebras B for which $k \otimes^L_{k[\epsilon]} B$ is derived Morita equivalent to A.

Received October 24, 2008; Revised March 15, 2009; Accepted March 19, 2009
Communicated by Prof. Dmitry Kaledin

We construct a canonical map μ_A from the set of equivalence classes of first-order Morita deformations to the second Hochschild cohomology group of A. We show that if A has bounded above cohomology, then μ_A is an injection. We also discuss some necessary, and some sufficient conditions for a Hochschild cocycle to represent a Morita deformation.

2 Some Background on dg Categories

2.1 The Hochschild complex

We work over a commutative ground ring k and assume everything to be k-linear.

For dg categories $\mathfrak{a}$ and $\mathfrak{b}$, the notation $\mathfrak{a} \otimes \mathfrak{b}$ will always be used for the *derived tensor product* (which is usually denoted by $\mathfrak{a} \otimes^L \mathfrak{b}$).

For the definition of the Hochschild complex of a dg category $\mathfrak{a}$, we refer the reader to [4]. Roughly speaking, the Hochschild complex $\mathbf{C}(\mathfrak{a})$ is the product double complex of the bicomplex with

$$\mathbf{C}^{*,n}(\mathfrak{a}) = \prod_{A_0,\dots,A_n} \mathrm{Hom}_k(\mathfrak{a}(A_{n-1}, A_n) \otimes \cdots \otimes \mathfrak{a}(A_0, A_1), \mathfrak{a}(A_0, A_n)),$$

and the familiar Hochschild differential. We adopt the sign in this differential of [8, Section 2.7].

The cohomology of the Hochschild complex has the following derived interpretation. Let $1_{\mathfrak{a}} = \mathfrak{a}(-,-)$ denote the $\mathfrak{a}$-$\mathfrak{a}$-bimodule with $1_{\mathfrak{a}}(A, A') = \mathfrak{a}(A, A')$ for $A, A' \in \mathfrak{a}$. For a triangulated category $\mathcal{T}$, let $\mathcal{T}^*$ denote the associated graded category with $\mathcal{T}^n(T, T') = \mathcal{T}(T, T'[n])$.

Lemma 2.1. Suppose $\mathfrak{a}$ is cofibrant over k (i.e $\mathfrak{a}$ has k-cofibrant Hom-modules). There is a canonical isomorphism

$$H^*\mathbf{C}(\mathfrak{a}) \cong D(\mathfrak{a}^{\mathrm{op}} \otimes \mathfrak{a})^*(1_{\mathfrak{a}}, 1_{\mathfrak{a}}).$$

Proof. This is contained in [4, Section 4.2].

2.2 The characteristic dg morphism

Let $\mathfrak{a}$ be a dg category. We consider the dg categories

$$\mathfrak{a} \subseteq \mathsf{tria}_{dg}(\mathfrak{a}) \subseteq \mathsf{per}_{dg}(\mathfrak{a}) \subseteq D_{dg}(\mathfrak{a})$$

that can all be constructed as quivers of twisted complexes over $\mathfrak{a}$ [8, Section 3.1]. With the notations of [8], we have $\mathsf{tria}_{dg}(\mathfrak{a}) = \mathsf{tw}_{\mathsf{ilnil}}(\mathfrak{a})_\infty$ and $D_{dg}(\mathfrak{a}) = \mathsf{Tw}_{\mathsf{ilnil}}(\mathfrak{a})_\infty$. The underlying triangulated categories are $H^0(D_{dg}(\mathfrak{a})) \cong D(\mathfrak{a})$, the derived category of $\mathfrak{a}$ (i.e. the quotient of the homotopy category of right dg $\mathfrak{a}$-modules by the subcategory of acyclic modules), and $H^0(\mathsf{tria}_{dg}(\mathfrak{a})) \cong \mathsf{tria}(\mathfrak{a})$, the smallest triangulated subcategory of $D(\mathfrak{a})$ containing $\mathfrak{a}$. In between we have $\mathsf{per}_{dg}(\mathfrak{a})$ with $H^0(\mathsf{per}_{dg}(\mathfrak{a})) \cong \mathsf{per}(\mathfrak{a})$, the closure of $\mathsf{tria}(\mathfrak{a})$ under direct summands, or, equivalently, the subcategory of compact objects in $D(\mathfrak{a})$.

In [8], the second author has constructed a B_∞-section of the canonical projection morphism $\mathbf{C}(D_{dg}(\mathfrak{a})) \longrightarrow \mathbf{C}(\mathfrak{a})$ (which is a morphism of B_∞-algebras) and considered the characteristic dg morphism

$$\bar{\chi} : \mathbf{C}(\mathfrak{a}) \longrightarrow \mathbf{C}(D_{dg}(\mathfrak{a})) \longrightarrow \prod_{M \in D_{dg}(\mathfrak{a})} D_{dg}(\mathfrak{a})(M, M).$$

The Mth component of $\bar{\chi}$ is given by

$$\bar{\chi}_M(\phi) = \sum_{n=0}^{\infty} (-1)^{\epsilon} \phi\big(\delta_M^{\otimes n}\big) \tag{1}$$

for $\phi \in \mathbf{C}(\mathfrak{a})$ and $M = (M, \delta_M) \in D_{dg}(\mathfrak{a})$. Here (M, δ_M) represents a free $\mathfrak{a}$-module (M, d) with "twisted" differential $d + \delta_M$. For the details and the signs in the expression, we refer the reader to [8, Proposition 3.11].

The map $\bar{\chi}$ is compatible with the derived interpretation of Hochschild cohomology, as the following proposition shows. Put $\chi_M = H^*(\bar{\chi}_M)$. Consider, for dg categories $\mathfrak{a}$, $\mathfrak{b}$, and $\mathfrak{c}$, the derived tensor functor

$$- \otimes^L_{\mathfrak{a}} - : D(\mathfrak{b}^{\mathrm{op}} \otimes \mathfrak{a}) \times D(\mathfrak{a}^{\mathrm{op}} \otimes \mathfrak{c}) \longrightarrow D(\mathfrak{b}^{\mathrm{op}} \otimes \mathfrak{c}).$$

Proposition 2.2. Suppose $\mathfrak{a}$ is cofibrant over k. The following diagram, in which the vertical arrows are the canonical isomorphisms, commutes:

$$\begin{tikzcd}
H^*\mathbf{C}(\mathfrak{a}) \arrow[r, "\chi_M"] \arrow[d, "\cong"'] & H^* D_{dg}(\mathfrak{a})(M, M) \arrow[d, "\cong"] \\
D(\mathfrak{a}^{\mathrm{op}} \otimes \mathfrak{a})^*(\mathfrak{a}, \mathfrak{a}) \arrow[r, "M\otimes^L -"'] & D(\mathfrak{a})^*(M, M)
\end{tikzcd}$$

Here we use the same notation $M \otimes^L -$ for the maps between graded Hom-modules that are induced by the functor $M \otimes^L - : D(\mathfrak{a}^{\mathrm{op}} \otimes \mathfrak{a})^* \longrightarrow D(\mathfrak{a})^*$ between graded categories.

Proof. By Proposition 2.3 applied to $\mathfrak{a} \longrightarrow D_{dg}(\mathfrak{a})$, the morphism

$$H^*\mathbf{C}(\mathfrak{a}) \longrightarrow H^*\mathbf{C}(D_{dg}(\mathfrak{a})) \xrightarrow[M\otimes^L_{D_{dg}(\mathfrak{a})}-]{} H^*D_{dg}(\mathfrak{a})(M,M)$$

is equal to $M \otimes^L_{\mathfrak{a}} -$. Hence, it suffices to show that, for any dg category $\mathfrak{a}$ and $A \in \mathfrak{a}$, the projection on the first column $\pi_A : \mathbf{C}(\mathfrak{a}) \longrightarrow \mathfrak{a}(A,A)$ satisfies $H^*(\pi_A) = A \otimes^L_{\mathfrak{a}} -$. To see this, let

$$BR(\mathfrak{a}) = \oplus_{A',A''}\mathfrak{a}(A'',-) \otimes B(\mathfrak{a})(A',A'') \otimes \mathfrak{a}(-,A') \longrightarrow \mathfrak{a}(-,-)$$

be the bar resolution of $\mathfrak{a}$ as an $\mathfrak{a}$-$\mathfrak{a}$-bimodule. For $\mathfrak{a}(-,A) \in D_{dg}(\mathfrak{a})$, a section ι_A of

$$BR(A) = \oplus_{A',A''}\mathfrak{a}(A'',A) \otimes B(\mathfrak{a})(A',A'') \otimes \mathfrak{a}(-,A') \longrightarrow \mathfrak{a}(-,A)$$

is determined by

$$\iota_A(1_A) = 1_A \otimes 1_k \otimes 1_A \in \mathfrak{a}(A,A) \otimes k \otimes \mathfrak{a}(A,A).$$

Then $A \otimes^L_{\mathfrak{a}} - = H^*(\psi)$, where ψ is the composition

$$\mathrm{Hom}_{\mathfrak{a}^{\mathrm{op}}\otimes\mathfrak{a}}(BR(\mathfrak{a}),\mathfrak{a}) \xrightarrow[A\otimes-]{} \mathrm{Hom}_{\mathfrak{a}}(BR(A),A) \xrightarrow[-\iota_A]{} \mathrm{Hom}_{\mathfrak{a}}(A,A).$$

Clearly, ψ is canonically isomorphic to π_A, which finishes the proof.

A dg functor $F : \mathfrak{a} \longrightarrow \mathfrak{b}$ is called quasi-fully faithful if for every $A, A' \in \mathfrak{a}$ the canonical $\mathfrak{a}(A,A') \longrightarrow \mathfrak{b}(F(A),F(A'))$ is a quasi-isomorphism. The symbol F will also denote any of the induced morphisms $D(\mathfrak{a}) \longrightarrow D(\mathfrak{b})$, $\mathfrak{a}^{\mathrm{op}} \otimes \mathfrak{a} \longrightarrow \mathfrak{b}^{\mathrm{op}} \otimes \mathfrak{b}$, $D(\mathfrak{a}^{\mathrm{op}} \otimes \mathfrak{a}) \longrightarrow D(\mathfrak{b}^{\mathrm{op}} \otimes \mathfrak{b})$.

Proposition 2.3. Let $F : \mathfrak{a} \longrightarrow \mathfrak{b}$ be a quasi-fully faithful functor between k-cofibrant dg categories.

1. For the canonical morphism $\phi_F : \mathbf{C}(\mathfrak{b}) \longrightarrow \mathbf{C}(\mathfrak{a})$, the cohomology $H^*(\phi_F)$ is given by

$$D(\mathfrak{b}\otimes\mathfrak{b}^{\mathrm{op}})^*(\mathfrak{b},\mathfrak{b}) \xrightarrow[F(\mathfrak{a})\otimes^L-]{} D(\mathfrak{b}\otimes\mathfrak{b}^{\mathrm{op}})^*(F(\mathfrak{a}),F(\mathfrak{a})) \xrightarrow[F^{-1}_{\mathfrak{a},\mathfrak{a}}]{} D(\mathfrak{a}\otimes\mathfrak{a}^{\mathrm{op}})^*(\mathfrak{a},\mathfrak{a}).$$

2. For $M \in D(\mathfrak{a})$, there is a commutative diagram

$$\begin{tikzcd}
D(\mathfrak{b}\otimes\mathfrak{b}^{\mathrm{op}})^*(\mathfrak{b},\mathfrak{b}) \arrow[r, "H^*(\phi_F)"] \arrow[d, "F(M)\otimes^L-"'] & D(\mathfrak{a}\otimes\mathfrak{a}^{\mathrm{op}})^*(\mathfrak{a},\mathfrak{a}) \arrow[d, "M\otimes^L-"] \\
D(\mathfrak{b})^*(F(M),F(M)) \arrow[r, "F^{-1}_{M,M}"'] & D(\mathfrak{a})^*(M,M).
\end{tikzcd}$$

Proof. (1) is contained in [4]. For (2), it is easy to see that the diagram

$$\begin{tikzcd}
D(\mathfrak{b}\otimes\mathfrak{b}^{\mathrm{op}})^*(\mathfrak{b},\mathfrak{b}) \arrow[r, "F(\mathfrak{a})\otimes^L -"] \arrow[d, "F(M)\otimes^L -"] & D(\mathfrak{b}\otimes\mathfrak{b}^{\mathrm{op}})^*(F(\mathfrak{a}),F(\mathfrak{a})) \arrow[d, "F(M)\otimes^L -"] & D(\mathfrak{a}\otimes\mathfrak{a}^{\mathrm{op}})^*(\mathfrak{a},\mathfrak{a}) \arrow[l, "F_{\mathfrak{a},\mathfrak{a}}"'] \arrow[d, "M\otimes^L -"] \\
D(\mathfrak{b})^*(F(M),F(M)) \arrow[r, "="'] & D(\mathfrak{b})^*(F(M),F(M)) & D(\mathfrak{a})^*(M,M) \arrow[l, "F_{M,M}"]
\end{tikzcd}$$

commutes.

3 Morita Deformations

3.1 $A_{[0,\infty[}$-deformations

Let $\mathfrak{a}$ be a k-linear dg category (or, more generally, an $A_{[0,\infty[}$-category). It follows from inspection of the Hochschild complex that Hochschild cohomology parametrizes first-order $A_{[0,\infty[}$-deformations [8, Section 4.5].

Proposition 3.1. There is a bijection

$$\nu_{\mathfrak{a}} : \mathrm{Ob}(A_{[0,\infty[}\mathsf{def}_{\mathfrak{a}}) \longrightarrow Z^2\mathbf{C}(\mathfrak{a}),$$

which descends to a bijection $\mathrm{Sk}(A_{[0,\infty[}\mathsf{def}_{\mathfrak{a}}) \longrightarrow H^2\mathbf{C}(\mathfrak{a})$. Here $A_{[0,\infty]}\mathsf{def}_{\mathfrak{a}}$ is the natural groupoid of first-order $A_{[0,\infty]}$-deformations of $\mathfrak{a}$, and $\mathrm{Sk}(A_{[0,\infty[}\mathsf{def}_{\mathfrak{a}})$ is its skeleton.

Proof. This is contained in [8, Proposition 4.11].

3.2 Morita deformations

Let dgcat_k be the category of small k-linear dg categories. Let Mo be the class of Morita morphisms, i.e. morphisms inducing a derived equivalence. In [12] and [13], a model structure on dgcat_k is constructed such that the homotopy category is $\mathsf{Hmo}_k = \mathsf{dgcat}_k[Mo^{-1}]$. Let

$$k\otimes^L - : \mathsf{Hmo}_{k[\epsilon]} \longrightarrow \mathsf{Hmo}_k$$

be the derived functor of $k\otimes - : \mathsf{dgcat}_{k[\epsilon]} \longrightarrow \mathsf{dgcat}_k$.

Definition 3.2. For a k-dg category $\mathfrak{a}$, a *Morita $k[\epsilon]$-deformation* of $\mathfrak{a}$ is a lift of $\mathfrak{a}$ along $k\otimes^L - : \mathsf{Hmo}_{k[\epsilon]} \longrightarrow \mathsf{Hmo}_k$. The natural groupoid of Morita deformations of $\mathfrak{a}$ is denoted by $\mathrm{hmodef}_{\mathfrak{a}} = \mathrm{hmodef}_{\mathfrak{a}}(k[\epsilon])$. Its objects are pairs $(\mathfrak{b},\varphi)$ with $\mathfrak{b}\in\mathsf{Hmo}_{k[\epsilon]}$ and

$\varphi : k \otimes^L \mathfrak{b} \longrightarrow \mathfrak{a}$ an isomorphism in Hmo_k. A morphism from $(\mathfrak{b}_1, \varphi_1)$ to $(\mathfrak{b}_2, \varphi_2)$ is an isomorphism $f : \mathfrak{b}_1 \longrightarrow \mathfrak{b}_2$ in $\mathsf{Hmo}_{k[\epsilon]}$ with $\varphi_2(k \otimes^L f) = \varphi_1$.

Our proofs will make intensive use of the *arrow category* $\mathfrak{c}_X$ associated to a bimodule $X \in C(\mathfrak{a}^{\mathrm{op}} \otimes \mathfrak{b})$ between dg categories $\mathfrak{b}$ and $\mathfrak{a}$, which has been introduced in [4, Section 4.5]. This dg category $\mathfrak{c}_X$ has $\mathrm{Ob}(\mathfrak{c}_X) = \mathrm{Ob}(\mathfrak{a}) \coprod \mathrm{Ob}(\mathfrak{b})$ and

$$\mathfrak{c}_X(A, A') = \mathfrak{a}(A, A'), \mathfrak{c}_X(B, B') = \mathfrak{b}(B, B), \mathfrak{c}_X(B, A) = X(B, A), \mathfrak{c}_X(A, B) = 0$$

for all objects $A, A' \in \mathfrak{a}$ and $B, B' \in \mathfrak{b}$.

Proposition 3.3. There is a canonical map

$$\mu_{\mathfrak{a}} : \mathrm{Sk}(\mathrm{hmodef}_{\mathfrak{a}}) \longrightarrow H^2\mathbf{C}(\mathfrak{a}),$$

where $\mathrm{Sk}(\mathrm{hmodef}_{\mathfrak{a}})$ is the skeleton of the deformation groupoid.

Proof. A Morita deformation of $\mathfrak{a}$ can be represented by a $k[\epsilon]$-cofibrant dg category $\bar{\mathfrak{b}}$ with $k \otimes \bar{\mathfrak{b}} = \mathfrak{b}$ together with a Morita bimodule $X \in C(\mathfrak{a}^{\mathrm{op}} \otimes \mathfrak{b})$ establishing an isomorphism $\mathfrak{b} \longrightarrow \mathfrak{a}$ in Hmo_k and thus an isomorphism $\phi_X : \mathbf{C}(\mathfrak{b}) \longrightarrow \mathbf{C}(\mathfrak{a})$ in $\mathsf{Ho}(B_\infty)$, the homotopy category of B_∞-algebras. Since $\bar{\mathfrak{b}}$ is an $A_{[0,\infty[}$-deformation of $\mathfrak{b}$, we put $\mu_{\mathfrak{a}}(\bar{\mathfrak{b}}) = H^2(\phi_X)([\nu_{\mathfrak{b}}(\bar{\mathfrak{b}})])$ with $\nu_{\mathfrak{b}}$ as in Proposition 3.1. To see that $\mu_{\mathfrak{a}}$ is well defined, let $(\bar{\mathfrak{c}}, Y \in C(\mathfrak{a}^{\mathrm{op}} \otimes \mathfrak{c}))$ be an equivalent Morita deformation. Consequently, there is a cofibrant Morita bimodule $\bar{Z} \in C(\bar{\mathfrak{b}}^{\mathrm{op}} \otimes \bar{\mathfrak{c}})$ with $k \otimes \bar{Z} = Z$ such that $X \circ Z = Y$ and thus $\phi_X \phi_Z = \phi_Y$. Hence, we are to show that $H^2(\phi_Z)([\nu_{\mathfrak{c}}(\bar{\mathfrak{c}})]) = [\nu_{\mathfrak{b}}(\bar{\mathfrak{b}})]$. Let $\mathfrak{c}_{\bar{Z}}$ and $\mathfrak{c}_Z$ denote the arrow categories of $\bar{Z}$ and Z. Then $\mathfrak{c}_{\bar{Z}}$ is an $A_{[0,\infty[}$-deformation of $\mathfrak{c}_Z$ and ϕ_Z is represented by $\mathbf{C}(\mathfrak{c}) \longleftarrow \mathbf{C}(\mathfrak{c}_Z) \longrightarrow \mathbf{C}(\mathfrak{b})$. Since $\nu_{\mathfrak{c}_Z}(\mathfrak{c}_{\bar{Z}}) \in \mathbf{C}^2(\mathfrak{c}_Z)$ clearly gets mapped to $\nu_{\mathfrak{c}}(\bar{\mathfrak{c}})$ on the left and to $\nu_{\mathfrak{b}}(\bar{\mathfrak{b}})$ on the right, this finishes the proof.

3.3 Injectivity of $\mu_{\mathfrak{a}}$

Remarkably, in order to be able to show that $\mu_{\mathfrak{a}}$ is injective, we already need some condition on $\mathfrak{a}$.

Definition 3.4.

1. A cochain complex M has *bounded above cohomology* if there is an $n_0 \in \mathbb{N}$ with $H^n M = 0$ for $n \geq n_0$.

2. A dg module $F \in C(\mathfrak{a})$ has *bounded above cohomology* if for every $A \in \mathfrak{a}$, the cochain complex $F(A)$ has bounded above cohomology.
3. A dg category $\mathfrak{a}$ has *bounded above cohomology* if the bimodule $1_{\mathfrak{a}} \in C(\mathfrak{a}^{\mathrm{op}} \otimes \mathfrak{a})$ has bounded above cohomology.

Lemma 3.5. Suppose $\mathfrak{a}$ and $\mathfrak{b}$ are dg categories and $X \in C(\mathfrak{a}^{\mathrm{op}} \otimes \mathfrak{b})$ is a Morita bimodule. If $\mathfrak{a}$ has bounded above cohomology, then the same holds for $\mathfrak{b}$ and X.

Proof. We may suppose that $\mathfrak{a}$, $\mathfrak{b}$, and X are cofibrant. The bimodule X defines a quasi-equivalence $\mathsf{per}_{dg}(\mathfrak{a}) \longmapsto \mathsf{per}_{dg}(\mathfrak{b}) : A \longmapsto X(-, A)$. Clearly, if $\mathsf{per}_{dg}(\mathfrak{a})$ has bounded above cohomology, then so has the quasi-equivalent $\mathsf{per}_{dg}(\mathfrak{b})$. Since $\mathsf{per}_{dg}(\mathfrak{b})(\mathfrak{b}(-, B), X(-, A)) = X(B, A)$ and $\mathsf{per}_{dg}(\mathfrak{b})(\mathfrak{b}(-, B), \mathfrak{b}(-, B')) = \mathfrak{b}(B, B')$, the claim follows once we show that $\mathsf{per}_{dg}(\mathfrak{a})$ has bounded above cohomology. The fact that $\mathsf{per}_{dg}(\mathfrak{a})$ has bounded above cohomology follows from the following observation. If $\mathcal{M}$ and $\mathcal{N}$ are two collections of dg $\mathfrak{a}$-modules such that $C_{dg}(\mathfrak{a})(M, N)$ has bounded above cohomology for $M \in \mathcal{M}$ and $N \in \mathcal{N}$, then we can close both $\mathcal{M}$ and $\mathcal{N}$ under shifts, cones, and summands without losing the property. Indeed, consider, for example, a map $m : M \longrightarrow M'$ in $\mathcal{M}$. Then $C_{dg}(\mathfrak{a})(\mathrm{cone}(m), N) = \mathrm{cone}(C_{dg}(M, N) \longrightarrow C_{dg}(M', N))$ still has bounded above cohomology.

Our interest in dg categories with bounded above cohomology comes from the following fact.

Lemma 3.6. Consider a cochain complex $\bar{M}$ of free $k[\epsilon]$-modules and put $M = k \otimes_{k[\epsilon]} \bar{M}$. If M has bounded above cohomology, then $\bar{M}$ is cofibrant over $k[\epsilon]$.

Proof. Since M has bounded above cohomology, M is quasi-isomorphic, hence homotopic over k, to a complex N with $N^n = 0$ for $n \geq n_0$ for some n_0. By the crude lifting lemma [7, Corollary 3.11], there exists a differential on the graded object $\bar{N} = k[\epsilon] \otimes_k N$ such that $\bar{N}$ and $\bar{M}$ become homotopic $k[\epsilon]$-modules. Now since $\bar{N}^n = 0$ for $n \geq n_0$, and $\bar{N}$ has $k[\epsilon]$-projective entries, it follows that $\bar{N}$ is $k[\epsilon]$-cofibrant, hence in particular homotopically projective as a $k[\epsilon]$-module. But then the homotopic $k[\epsilon]$-module $\bar{M}$ is also homotopically projective. Since it also has $k[\epsilon]$-projective entries, it follows that $\bar{M}$ is $k[\epsilon]$-cofibrant.

Proposition 3.7. If $\mathfrak{a}$ has bounded above cohomology, then $\mu_{\mathfrak{a}}$ is an injection.

Proof. Let $(\bar{\mathfrak{b}}, Z \in C(\mathfrak{a}^{\mathrm{op}} \otimes \mathfrak{b}))$ and $(\bar{\mathfrak{c}}, Y \in C(\mathfrak{a}^{\mathrm{op}} \otimes \mathfrak{c}))$ be Morita deformations of $\mathfrak{a}$ with $\mu_{\mathfrak{a}}(\bar{\mathfrak{b}}) = \mu_{\mathfrak{a}}(\bar{\mathfrak{c}})$. Let $X \in C(\mathfrak{b}^{\mathrm{op}} \otimes \mathfrak{c})$ be a cofibrant bimodule representing $Y^{-1}Z$ in Hmo_k and let $\mathfrak{c}_X$ be the corresponding arrow category. Let $\gamma \in Z^2(\mathfrak{c}_X)$ be an element of which the images under $\mathbf{C}(\mathfrak{c}) \longleftarrow \mathbf{C}(\mathfrak{c}_X) \longrightarrow \mathbf{C}(\mathfrak{b})$ represent $[\nu_{\mathfrak{c}}(\bar{\mathfrak{c}})]$ and $[\nu_{\mathfrak{b}}(\bar{\mathfrak{b}})]$. By Lemma 3.8, we may assume that the images of γ are precisely $\nu_{\mathfrak{c}}(\bar{\mathfrak{c}})$ and $\nu_{\mathfrak{b}}(\bar{\mathfrak{b}})$. Since in the zeroth column of the Hochschild bicomplex, the map $\mathbf{C}(\mathfrak{c}_X) \longrightarrow \mathbf{C}(\mathfrak{c}) \oplus \mathbf{C}(\mathfrak{b})$ induces an isomorphism, we have in particular that $\gamma_0 = 0$. Let $\bar{\mathfrak{c}}_X = \nu_{\mathfrak{c}_X}^{-1}(\gamma)$ be the A_∞-deformation of $\mathfrak{c}_X$ corresponding to γ, and let $\bar{X} \in \mathsf{Mod}_\infty(\bar{\mathfrak{b}}^{\mathrm{op}} \otimes \bar{\mathfrak{c}})$ be the corresponding A_∞-bimodule. Using the equivalence $D(\bar{\mathfrak{b}}^{\mathrm{op}} \otimes \bar{\mathfrak{c}}) \longrightarrow D_\infty(\bar{\mathfrak{b}}^{\mathrm{op}} \otimes \bar{\mathfrak{c}})$ proved in Lemma 4.1.3.8 of [6] over fields but also valid over $k[\epsilon]$, we can replace $\bar{X}$ by a quasi-isomorphic cofibrant $\tilde{X} \in C(\bar{\mathfrak{b}}^{\mathrm{op}} \otimes \bar{\mathfrak{c}})$. By Lemmas 3.5 and 3.6, $\tilde{X}$ is $k[\epsilon]$-cofibrant and, consequently, $k \otimes_{k[\epsilon]} \tilde{X}$ is a Morita bimodule quasi-isomorphic to X. By Lemma 3.9, $\tilde{X}$ is a Morita bimodule; hence, it constitutes an equivalence of Morita deformations between $\bar{\mathfrak{b}}$ and $\bar{\mathfrak{c}}$.

Lemma 3.8. Let $p_i : M \longrightarrow M_i$, $i = 1, \ldots, n$ be morphisms of chain complexes with graded sections $s_i : M_i \longrightarrow M$ such that $p_i s_j = 0$ unless $i = j$. For some k, consider elements $m_i \in M_i^k$ for $i = 1, \ldots, n$. If there exists an $m \in M^k$ with $H^k(p_i)([m]) = [m_i]$ for every i, then there exists an m' with $p_i(m') = m_i$ for every i.

Proof. If $m_i = p_i(m) + d(n_i)$, put $m' = m + \sum_{i=1}^n d(s_i(n_i))$.

Lemma 3.9. Let $\bar{\mathfrak{a}}$ and $\bar{\mathfrak{b}}$ be $k[\epsilon]$-cofibrant dg categories, $\bar{X} \in C(\bar{\mathfrak{a}}^{\mathrm{op}} \otimes \bar{\mathfrak{b}})$ be a cofibrant bimodule, and put $\mathfrak{a} = k \otimes_{k[\epsilon]} \bar{\mathfrak{a}}$, $\mathfrak{b} = k \otimes_{k[\epsilon]} \bar{\mathfrak{b}}$, $X = k \otimes_{k[\epsilon]} \bar{X}$. If X is a Morita bimodule, then so is $\bar{X}$.

Proof. For every $A \in \mathfrak{a}$, we consider the cofibrant $\bar{\mathfrak{b}}$-module $\bar{X}(-, A)$. By assumption, the objects $X(-, A) = k \otimes_{k[\epsilon]} \bar{X}(-, A)$ form a set of compact generators of $D(\mathfrak{b})$; hence, by Lemma 3.10, the objects $\bar{X}(-, A)$ form a set of compact generators of $D(\bar{\mathfrak{b}})$. It remains to show that for $A, A' \in \mathfrak{a}$, the canonical

$$\bar{\lambda}_{A,A'} : \bar{\mathfrak{a}}(A, A') \longrightarrow \mathrm{Hom}_{\bar{\mathfrak{b}}}(\bar{X}(-, A), \bar{X}(-, A'))$$

is a quasi-isomorphism. Since $\bar{X}(-, A)$ and $\bar{X}(-, A')$ are cofibrant, we have triangles

$$\triangle_{\bar{X}(-,A')} = X(-, A') \longrightarrow \bar{X}(-, A') \longrightarrow X(-, A') \longrightarrow$$

and $\mathrm{Hom}_{\bar{\mathfrak{b}}}(\bar{X}(-, A), \triangle_{\bar{X}(-,A')})$. Since $\bar{\mathfrak{a}}$ is $k[\epsilon]$-cofibrant, we have a triangle

$$\triangle_{\bar{\mathfrak{a}}(A,A')} : \mathfrak{a}(A, A') \longrightarrow \bar{\mathfrak{a}}(A, A') \longrightarrow \mathfrak{a}(A, A') \longrightarrow .$$

We have a morphism of triangles

$$\triangle_{\bar{\mathfrak{a}}(A,A')} \longrightarrow \mathrm{Hom}_{\bar{\mathfrak{b}}}\big(\bar{X}(-,A), \triangle_{\bar{X}(-,A')}\big)$$

in which the middle arrow is given by $\bar{\lambda}_{A,A'}$ and the other two arrows by the canonical $\lambda_{A,A'} : \mathfrak{a}(A, A') \longrightarrow \mathrm{Hom}_{\mathfrak{b}}(X(-, A), X(-, A'))$. This finishes the proof.

Lemma 3.10. Let $\bar{\mathfrak{a}}$ be a $k[\epsilon]$-cofibrant dg category and put $\mathfrak{a} = k \otimes_{k[\epsilon]} \bar{\mathfrak{a}}$. Consider a set of cofibrant objects $\bar{X}_i \in C(\bar{\mathfrak{a}})$ and put $X_i = k \otimes_{k[\epsilon]} \bar{X}_i$.

1. The objects $\bar{X}_i$ generate $D(\bar{\mathfrak{a}})$ if and only if the objects X_i generate $D(\mathfrak{a})$.
2. $\bar{X}_i$ is compact in $D(\bar{\mathfrak{a}})$ if and only if X_i is compact in $D(\mathfrak{a})$.

Proof. For $M \in C(\mathfrak{a})$, we have $\mathrm{Hom}_{\bar{\mathfrak{a}}}(\bar{X}_i, M) = \mathrm{Hom}_{\mathfrak{a}}(X_i, M)$. This shows the necessity in (1). For any cofibrant $\bar{M} \in C(\bar{\mathfrak{a}})$ with $M = k \otimes_{k[\epsilon]} \bar{M}$, we have a triangle

$$\triangle_{\bar{M}} = M \longrightarrow \bar{M} \longrightarrow M \longrightarrow$$

in $D(\bar{\mathfrak{a}})$. Since $\bar{X}_i$ is cofibrant, we obtain triangles in $D(k[\epsilon])$

$$\mathrm{Hom}_{\mathfrak{a}}(X_i, M) \longrightarrow \mathrm{Hom}_{\bar{\mathfrak{a}}}(\bar{X}_i, M) \longrightarrow \mathrm{Hom}_{\mathfrak{a}}(X_i, k \otimes M) \longrightarrow .$$

This already proves the sufficiency in (1). For (2), consider objects $\bar{M}_j \in C(\bar{\mathfrak{a}})$ with $M_j = k \otimes_{k[\epsilon]} \bar{M}_j$ and the canonical

$$\bar{\lambda} : \coprod_j \mathrm{Hom}_{\bar{\mathfrak{a}}}(\bar{X}_i, \bar{M}_j) \longrightarrow \mathrm{Hom}_{\bar{\mathfrak{a}}}\left(\bar{X}_i, \coprod_j \bar{M}_j\right).$$

For $\bar{M}_j = M_j \in C(\mathfrak{a})$, $\bar{\lambda}$ coincides with the canonical $\lambda : \coprod_j \mathrm{Hom}_{\mathfrak{a}}(X_i, M_j) \longrightarrow \mathrm{Hom}_{\mathfrak{a}}(X_i, \coprod_j M_j)$. Now suppose the objects $\bar{M}_j$ are cofibrant in $C(\bar{\mathfrak{a}})$. We obtain a morphism of triangles

$$\coprod_j \mathrm{Hom}_{\bar{\mathfrak{a}}}(\bar{X}_i, \triangle_{\bar{M}_j}) \longrightarrow \mathrm{Hom}_{\bar{\mathfrak{a}}}\left(\bar{X}_i, \coprod_j \triangle_{\bar{M}_j}\right),$$

which finishes the proof.

3.4 Image of $\mu_{\mathfrak{a}}$

We start this section by giving a very general description of the image of $\mu_{\mathfrak{a}}$. This description involves the map from $H^*\mathbf{C}(\mathfrak{a})$ into the center $Z^*D(\mathfrak{a})$ of the derived category of $\mathfrak{a}$. We restrict our attention to $\mathsf{per}_{dg}(\mathfrak{a}) \subseteq D_{dg}(\mathfrak{a})$. For every $M \in \mathsf{per}_{dg}(\mathfrak{a})$, we havea morphism of complexes

$$\bar{\chi}_M : \mathbf{C}(\mathfrak{a}) \longrightarrow \mathbf{C}(\mathsf{per}_{dg}(\mathfrak{a})) \longrightarrow \mathsf{per}_{dg}(\mathfrak{a})(M, M)$$

and the induced

$$\chi_M : H^2\mathbf{C}(\mathfrak{a}) \longrightarrow \mathrm{Ext}^2_{\mathfrak{a}}(M, M).$$

Definition 3.11. We say that a full subcategory $\mathfrak{m} \subseteq \mathsf{per}(\mathfrak{a})$ *generates* $\mathsf{per}(\mathfrak{a})$ if the bimodule associated to the inclusion yields a quasi-equivalence $\mathsf{per}_{dg}(\mathfrak{m}) \cong \mathsf{per}_{dg}(\mathfrak{a})$.

Proposition 3.12. For $\phi \in H^2\mathbf{C}(\mathfrak{a})$, consider the following properties:

1. ϕ is in the image of $\mu_{\mathfrak{a}}$, and
2. there is a full generating subcategory $\mathfrak{m} \subseteq \mathsf{per}_{dg}(\mathfrak{a})$ such that $\chi_M(\phi) = 0$ for every $M \in \mathfrak{m}$.

Property (1) always implies (2). If $\mathfrak{a}$ has bounded above cohomology, then (1) and (2) are equivalent.

Proof. Suppose (1) holds. Let $(\bar{\mathfrak{b}}, X \in C(\mathfrak{a}^{\mathrm{op}} \otimes \mathfrak{b}))$ be a Morita deformation of $\mathfrak{a}$ with $\mu_{\mathfrak{a}}(\bar{\mathfrak{b}}) = \phi$. There is an induced map $\mathsf{per}_{dg}(\mathfrak{b}) \longrightarrow \mathsf{per}_{dg}(\mathfrak{a}) : \mathfrak{b}(-, B) \longmapsto X(B, -)$ identifying $\mathfrak{b}$, up to quasi-equivalence $\varphi : \mathfrak{b} \longrightarrow \mathfrak{m}$, with a full subcategory $\mathfrak{m}$ of $\mathsf{per}_{dg}(\mathfrak{a})$. By Proposition 2.3(2), for every $B \in \mathfrak{b}$, the canonical $\mathrm{Ext}^2_{\mathfrak{b}}(B, B) \longrightarrow \mathrm{Ext}^2_{\mathfrak{a}}(\varphi(B), \varphi(B))$ maps $\chi_B(\phi)$ to $\chi_{\varphi(B)}(\varphi(\phi))$. Since $\bar{\mathfrak{b}}$ is a dg deformation of $\mathfrak{b}$, we have $\chi_B(\phi) = 0$ and hence also $\chi_{\varphi(B)}(\varphi(\phi)) = 0$. This proves (2). Now suppose $\mathfrak{a}$ has bounded above cohomology and (2) holds. We can construct an A_∞ deformation $\bar{\mathfrak{m}}$ of $\mathfrak{m}$ with an underlying graded object $k[\epsilon] \otimes_k \mathfrak{m}$. By the assumption and Lemmas 3.5 and 3.6, $\bar{\mathfrak{m}}$ is $k[\epsilon]$-cofibrant. It then suffices to take a cofibrant dg category quasi-equivalent to $\bar{\mathfrak{m}}$; see Section 2.3.4 of [6] to obtain a Morita deformation of $\mathfrak{a}$. This finishes the proof.

From the previous proposition, we deduce the following restriction on the image of $\mu_{\mathfrak{a}}$.

Proposition 3.13. Consider $\phi \in H^2\mathbf{C}(\mathfrak{a})$.

1. The subcategory of objects M, for which ϕ_M is nilpotent, is closed under shifts, cones, and direct summands.
2. If ϕ is in the image of $\mu_{\mathfrak{a}}$, then ϕ_M is nilpotent for every $M \in \mathsf{per}(\mathfrak{a})$.

Proof. If ϕ is in the image of $\mu_{\mathfrak{a}}$, then by Proposition 3.12, there is a full generating subcategory $\mathfrak{m} \subseteq \mathsf{per}_{dg}(\mathfrak{a})$ with $\phi_M = 0$ for every $M \in \mathfrak{m}$. Consequently, (2) immediately follows from (1). In (1), all three properties follow from bifunctoriality of the derived tensor product $- \otimes^L_{\mathfrak{a}} - : D(\mathfrak{a})^* \otimes D(\mathfrak{a}^{\mathrm{op}} \otimes \mathfrak{a})^* \longrightarrow D(\mathfrak{a})^*$, more precisely, from the fact that a map $f : M \longrightarrow N$ in $D(\mathfrak{a})^*$ gives rise to a (super)commutative square

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\phi_M"'] & N \arrow[d, "\phi_N"] \\ M \arrow[r, "f"'] & N. \end{tikzcd}$$

First, note that we have $\phi_{M[n]} = \phi_M[n]$ and $\phi_{M\oplus N} = \phi_M \oplus \phi_N$. For the cone, suppose we have a triangle $M \longrightarrow P \longrightarrow N \longrightarrow$ and $\phi^m_M = 0$ and $\phi^n_N = 0$. Then we can draw a diagram

$$\begin{tikzcd} M \arrow[r] \arrow[d, "\phi^n_M"'] & P \arrow[r] \arrow[d, "\phi^n_P"] \arrow[dl] & N \arrow[d, "\phi^n_N=0"] \\ M \arrow[r] \arrow[d, "0=\phi^m_M"'] & P \arrow[r] \arrow[d, "\phi^m_P"'] & N \arrow[d, "\phi^m_N"] \arrow[dl] \\ M \arrow[r] & P \arrow[r] & N \end{tikzcd}$$

showing that $\phi^{n+m}_P = 0$.

There are, of course, plenty of examples where the necessary condition of Proposition 3.13 is not fulfilled for some $\phi \in H^2\mathbf{C}(\mathfrak{a})$, the most notable being the “graded field” from section 5.4 of [5].

Example 3.14. Consider the graded algebra $A = k[u, u^{-1}]$, where u is of degree 2, endowed with the zero differential. In this case, $H^2\mathbf{C}(\mathfrak{a}) = k$ and for the Hochschild two-cocycle $u \in A^2$, $u_A \in \mathrm{Ext}^2_A(A, A) = A^2$ is an isomorphism, hence certainly not nilpotent. Consequently, there is no Morita deformation of A corresponding to u. Morally, this corresponds to the fact that a graded field should be rigid.

Of course, this situation cannot occur when $\mathfrak{a}$ has bounded above cohomology.

Proposition 3.15. If $\mathfrak{a}$ has bounded above cohomology, then for every $\phi \in H^2\mathbf{C}(\mathfrak{a})$ and $M \in \mathsf{per}(\mathfrak{a})$, $\phi_M \in D(\mathfrak{a})^2(M, M)$ is nilpotent.

Proof. If $\mathfrak{a}$ has bounded above cohomology, then so has $\mathsf{per}_{dg}(\mathfrak{a})$. Consequently, for $M \in \mathsf{per}_{dg}(\mathfrak{a})$, we have $D(\mathfrak{a})^n(M, M) = H^n\mathsf{per}_{dg}(\mathfrak{a})(M, M) = 0$ for $n \geq n_0$ for a certain $0 < n_0$, and hence $\phi_M^{n_0} = 0$.

Suppose from now on that $\phi \in Z\mathbf{C}^2(\mathfrak{a})$ is such that ϕ_A is nilpotent for every $A \in \mathfrak{a}$. According to Propositions 3.12 and 3.13, it makes sense to wonder whether ϕ is in the image of $\mu_{\mathfrak{a}}$. It was claimed in [2] and Section 5.4 of [5] that this was indeed the case, but further investigations have shown this conclusion to be premature. Proposition 3.12 tells us that we should look for a generating subcategory $\mathfrak{m} \subseteq \mathsf{per}_{dg}(\mathfrak{a})$ on which $\chi(\phi)$ vanishes. The following proposition describes a way of finding new generating subcategories of $\mathsf{per}(\mathfrak{a})$.

Proposition 3.16. Let $\mathfrak{a}$ be a dg category and suppose $\mathfrak{m}$ generates $\mathsf{per}(\mathfrak{a})$. Consider for a certain $M_0 \in \mathfrak{m}$ and $f \in \mathsf{per}(\mathfrak{a})^n(M, M)$ with $f^m = 0$ the object $\tilde{M}_0 = \mathrm{cone}(f)$ and the full subcategory $\tilde{\mathfrak{m}}$ spanned by $(\mathrm{Ob}(\mathfrak{m}) \setminus \{M_0\}) \cup \{\tilde{M}_0\}$. Then $\tilde{\mathfrak{m}}$ generates $\mathsf{per}(\mathfrak{a})$.

Proof. It suffices to show that M_0 belongs to the closure $\mathsf{per}(\tilde{\mathfrak{m}})$ of $\tilde{\mathfrak{m}}$ in $D(\mathfrak{a})$ under shifts, cones, and direct summands. By the octahedral axiom, $\mathrm{cone}(f^p)$ is in a triangle with $\mathrm{cone}(f)$ and $\mathrm{cone}(f^{p-1})$ and, hence, $\mathrm{cone}(f^p)$ belongs to $\mathsf{per}(\mathrm{cone}(f))$. Since $f^m = 0$, we have that $\mathrm{cone}(f^m) = M_0 \oplus M_0[mn]$ so M_0 belongs to $\mathsf{per}(\tilde{M}_0)$ hence to $\mathsf{per}(\tilde{\mathfrak{m}})$.

An obvious candidate for a new generating subcategory is $\tilde{\mathfrak{a}} \subseteq \mathsf{per}_{dg}(\mathfrak{a})$ in which each $A \in \mathfrak{a}$ has been replaced by the cone $\tilde{A}$ of

$$\phi_A \in \mathfrak{a}(A, A)^2.$$

In twisted object description, $\tilde{A}$ is given by $A \oplus \Sigma^{-1}A$ with

$$\delta_{\tilde{A}} = \begin{pmatrix} 0 & \sigma\phi_A \\ 0 & 0 \end{pmatrix}$$

for $\sigma\phi_A \in (\Sigma^{-1}\mathfrak{a}(A, A))^1$. If the dg structure on $\mathfrak{a}$ is given by $m + d$ (the sum of the multiplication and the differential, both considered as elements in the Hochschild complex), then the dg structure on $\mathsf{per}_{dg}(\mathfrak{a})$ is given by $m + d + m\{\delta\}$ (where $m\{\delta\} = m \bullet \delta$ is the dot product or first brace operation), and

$$\bar{\chi}_{\tilde{A}}(\phi) = \phi_A - \phi_1(\delta_{\tilde{A}}) = \begin{pmatrix} \phi_A & \sigma\phi_1(\phi_A) \\ 0 & \phi_A \end{pmatrix}$$

according to (1). Here, ϕ_1 is the component of ϕ of arity 1. In dg module description, $\tilde{A}$ corresponds to the cone of $\sigma^2\phi_A : \Sigma^{-2}A \longrightarrow A$.

Proposition 3.17. We have $0 = \chi_{\tilde{A}}(\phi) \in D(\mathfrak{a})(\tilde{A}, \Sigma^2\tilde{A})$ if and only if

$$0 = -\phi_1(\delta_{\tilde{A}}) = \begin{pmatrix} 0 & \sigma\phi_1(\phi_0) \\ 0 & 0 \end{pmatrix} \in H^2\mathsf{per}_{dg}(\mathfrak{a})(\tilde{A}, \tilde{A}).$$

Proof. This immediately follows from Lemma 3.18.

Lemma 3.18. Let $\mathfrak{a}$ be a dg category, M be a dg module, and $f : \Sigma^{-n}M \longrightarrow M$ be a map in $C(\mathfrak{a})$. The map

$$\begin{pmatrix} \Sigma^n f & 0 \\ 0 & \Sigma f \end{pmatrix} : \mathrm{cone}(f) \longrightarrow \mathrm{cone}(\Sigma^n f)$$

is nullhomotopic.

Proof. A nullhomotopy is given by $\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$.

Unfortunately, the condition in Proposition 3.17 is not always fulfilled, as the following example shows.

Example 3.19. Let $B = k\langle u, v\rangle$ be the free graded algebra on generators u in degree 2 and v in degree 3, and let $A = k[u, v] = k\langle u, v\rangle/(uv - vu, v^2)$ be the free supercommutative graded algebra on u and v. We equip both algebras with the zero differential. Then for A, we can define a Hochschild two-cocycle $\phi = \phi_0 + \phi_1$ with $\phi_0 = u$ and ϕ_1 equal to the derivation with $\phi_1(u) = v$ and $\phi_1(v) = 0$. Then, clearly, $-\phi_1(\delta_{\tilde{A}})$ cannot be written as a boundary.

The question whether it is possible to construct nonzero objects M in $\mathsf{tria}(\mathfrak{a})$, $\mathsf{per}(\mathfrak{a})$, or even $D(\mathfrak{a})$ for which $\chi_M(\phi)$ vanishes seems to lead to combinatorial puzzles that we have not been able to solve so far. Notice, however, that for dg-derived categories of abelian categories, these problems do not arise; see [7] for the general framework and [14] for a class of examples. On the other hand, at least in a topological context, the above cone construction does not allow one to construct objects where a given element of the center of the category acts by 0, as shown in [11].

Of course, without conditions on $\mathfrak{a}$, such objects need not exist, as Example 3.14 shows. Indeed, in this case Proposition 3.17 does apply, but $\tilde{A}$ is zero. Clearly, on every nonzero object M, $\bar{\chi}_M(\phi) = \phi_A$ is an isomorphism, hence nonzero.

In the case where $\mathfrak{a}$ has bounded above cohomology, it would certainly be desirable to obtain a better understanding of the locus of objects M with vanishing $\chi_M(\phi)$ (note that the problem in Example 3.19 persists even if we make the algebra bounded above by considering some further quotient). This, and the question whether it is possible to find perfect generators with vanishing $\chi_M(\phi)$ for classes of dg algebras of particular interest, like smooth proper dg algebras, remains the topic of work in progress.

Acknowledgment

The second-named author is a postdoctoral fellow with the Fund of Scientific Research Flanders (FWO). Both authors thank an anonymous referee for his comments on a previous version of this paper.

References

[1] Anel, M. “Moduli of linear and abelian categories.” (2006): preprint arXiv:math/0607385.

[2] Geiss, C., and B. Keller. “Infinitesimal deformations of derived categories.” *Oberwolfach Report* 6 (2005): 388–9.

[3] Gerstenhaber, M. “On the deformation of rings and algebras.” *Annals of Mathematics* 79, no. 2 (1964): 59–103.

[4] Keller, B. “Derived invariance of higher structures on the Hochschild complex.” preprint available at: math.jussieu.fr/˜keller/publ/dih.dvi.

[5] Keller, B. “On differential graded categories.” In *Proceedings of the International Congress of Mathematicians,* 151–90. Vol. 2. Zürich (Switzerland): European Mathematical Society, 2006.

[6] Lefèvre-Hasegawa, K. “Sur les A_∞-catégories.” (2003): preprint arXiv:math.CT/0310337.

[7] Lowen, W. “Obstruction theory for objects in abelian and derived categories.” *Communications in Algebra* 33, no. 9 (2005): 3195–223.

[8] Lowen, W. “Hochschild cohomology, the characteristic morphism and derived deformations.” *Compositio Mathematica* 144, no. 6 (2008): 1557–80.

[9] Lowen, W., and M. Van den Bergh. “Hochschild cohomology of abelian categories and ringed spaces.” *Advances in Mathematics* 198, no. 1 (2005): 172–221.

[10] Nicolás, P. “The bar derived category of a curved dg algebra.” *Journal of Pure and Applied Algebra* 212, no. 12 (2008): 2633–59.

[11] Schwede, S. “Algebraic versus topological triangulated categories.” (2008): preprint arXiv:0807.2592.

[12] Tabuada, G. “Invariants additifs de DG-catégories.” *International Mathematics Research Notices* 53 (2005): 3309–39.

[13] Tabuada, G. Erratum to: “Additive invariants of DG-categories” [French]. *International Mathematics Research Notices* 24 (2007): 1–17.

[14] Toda, Y. “Deformations and Fourier–Mukai transforms.” (2005): preprint arXiv:math/0502571.